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Mathematical Model for Managing the Renewable Resources

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ABSTRACT: The demand for the natural resources is continuously increasing due to growth in human population. One of the important natural resource that feeds millions of people is fisheries and is currently under great stress and needs to be managed in sustainable way, if this resource is not managed properly it will cause widespread starvation, species extinction, and considerable human instability. The present study is an attempt to introduce some of the mathematical models that may be used to understand the natural resources like fisheries that will aid in proper management.

Keyword: Constant harvesting, Proportional harvesting, MSH.

I. INTRODUCTION

Proper management and planning of natural resources and environmental systems has become an issue from last few years as pollution and resource stress problems have led to a variety of impacts and liabilities. On the other hand, due to comples problems of world, achieving a reasonable and efficient management strategy is difficult as different many conflicting factors have to be balanced. There are many factors that need to be considered by planners and decision-makers, such as social, economic, technical, legislational, institutional, and political issues, as well as environmental protection and resources conservation in resources and environmental systems. In addition to that, variety of processes and activities are interrelated to each other, resulting in complicated systems with interactive, dynamic, nonlinear, multiobjective, multistage, multilayer, and uncertain features. These complexities may increase due to their interactions with economic consequences when the promises of targets are violated. Mathematical models are effective tools that could help examine economic, environmental, and ecological impacts of alternative pollution-control and resources-conservation actions, and thus aid planners or decision-makers in formulating cost-effective management policies.

For sustaining life, dynamics of each food resources like animals, fish, plants etc takes steps to ensure that overharvesting, pollution, and urban sprawl do not drive these resources into collapse. Hence managers determine maximum harvest levels that are sustainable over the long run called *maximum sustainable harvest* (*MSH*) with minimum effort without driving a particular food source to extinction.

A simple logistic model with the inclusion of a harvesting contribution by taking input from Beddington and May is described [2]. Although it is a particularly simple one it brings out several interesting and important points which are more sophisticated models must also take into account. The economic factors by Clark [6,7,8] and harvest models with optimal control theory by Kot [11], a review by Plant and Mangel [14] are all concerned with insect pest management. Rotenberg [15] considered the logistic model with harvesting, with a view [1,12,13] to making the model more quantitative. He also examined the effects of certain stochastic parameters on possible population extinction.

When a species is introduced and no harvesting occurs, it will grow to a size that is nearly constant and regulated by the carrying capacity K of the environment. At this point, the rate of change of the population is nearly zero, as the birth and death rates are equal, predictable, and stable. When harvesting occurs, population levels can change drastically, depending on the particular level of harvest.

On the other hand, it is possible in some populations to harvest a certain amount of that resource with the knowledge that new births will exceed deaths and push the population back to its carrying capacity. It turns out that we can use the concept of differentiation to estimate maximum sustainable harvest.

II. OBJECTIVES

1) To determine the population function in proportional and constant harvesting.
2) To estimate the Maximum Sustainable Harvesting.
3) To know equilibrium state in the population during harvesting and time taken to reach it.
4) Determination of an optimal harvesting strategy by steady state analysis of the dynamic behavior.

III. EVALUATION

To see how this model works, let we assumed that a population grows according to the logistic model, written as a derivative. In this case, the mathematical model can be written as

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \qquad P(0) = P_0 \qquad (1)$$

Where *r* is a constant, *P* is the population at time *t*, and *K* is the carrying capacity of the environment. By solving this equation for population *P*, we get

$$P = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}} \qquad (2)$$

As $t \to \infty$, $P_{max} = K$: Carrying Capacity, this is shown in figure 1.

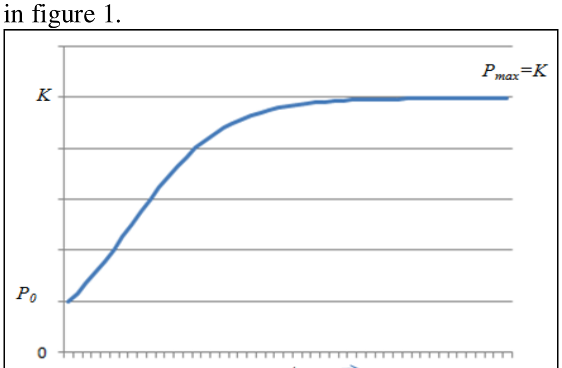

Fig. 1

The time required to attain a population is given by

$$t = \frac{1}{r}\ \ln\left[\frac{P(K - P_0)}{P_0(K - P)}\right] \quad \forall\ K \neq P \qquad (3)$$

IV. PROPORTIONAL HARVESTING

When harvesting occurs and some proportion of the population is removed, our logistic model changes to

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P - sP = f(P), \ \ P(0) = P_0 \ \ (4)$$

Where *r* is a population growth rate constant, *s* is the fraction of the population that is harvested, *P* is the population at time *t*, and *K* is the carrying capacity of the environment. By solving this equation for population *P*, we get

$$P = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-(r-s)t}} \qquad (5)$$

As $t \to \infty$, $P_{max} = K$: Carrying Capacity, this is shown in figure 2.

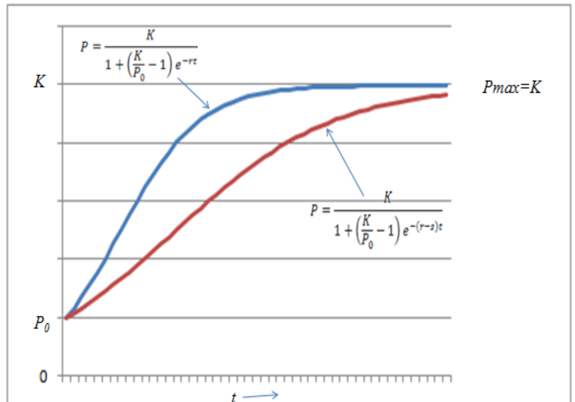

Fig. 2

The time required to attain a population in proportional harvesting is given by

$$t = \frac{1}{(r-s)}\ \ln\left[\frac{P(K - P_0)}{P_0(K - P)}\right] \quad \forall\ K \neq P \qquad (6)$$

A population is referred to as being in *equilibrium* when its rate of change is 0. Notice that in Equation (4), the population has reached a state of equilibrium when

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P - sP = 0$$

By solving, we get

$$P_e = 0 \ \ OR \ \ P_e = K\left(1 - \frac{s}{r}\right) \qquad (7)$$

The population reaches to the equilibrium state (*Pe*) after time *Te*,

$$T_e = \frac{1}{(r-s)}\ln\left[\frac{P_e(K - P_0)}{P_0(K - P_e)}\right] \qquad (8)$$

OR

$$T_e = \frac{1}{(r-s)}\ \ln\left[\left(1 - \frac{K}{P_0}\right)\left(1 - \frac{r}{s}\right)\right] \qquad (9)$$

This implies that at equilibrium, the proportion of the population that is harvested can be determined by multiplying both sides of Equation (7) by *s*, we get,

$$P_h = sP_e = sK\left(1 - \frac{s}{r}\right) \qquad (10)$$

Note that *sP* will be nonnegative when $\frac{s}{r} < 1$, or when $s < r$ that is, when the proportion of the population harvested is less than the rate of growth of the population. Observe that if $r > s$, the population will eventually disappear.

We can use Equation (10) to determine maximum sustainable harvest by finding when $P_h = sP_e$ reaches its maximum level. Note that the right-hand side of Equation (10) is a quadratic function of s, which attains a maximum value when $s = r/2$. Thus, the maximum sustainable harvest for proportional harvesting is found by substituting $s = r/2$ into Equation (10). That is,

$$P_M = MSH = P_e.\frac{r}{2} = \frac{r}{2}K\left(1 - \frac{r/2}{r}\right) = \frac{rK}{4} \qquad (11)$$

Now steady state analysis of the dynamic behavior tells us something different from the usual.
The growth rate $f\,(P)$ in (4) as a function of P for various efforts s. Linearising (4) about P_e gives

$$\frac{d(P - P_e)}{dt} \cong (P - P_e)f'(P_e) = (s - r)(P - P_e) \; (12)$$

which shows linear stability if $s < r$: arrows indicate stability or instability in Figure 3.

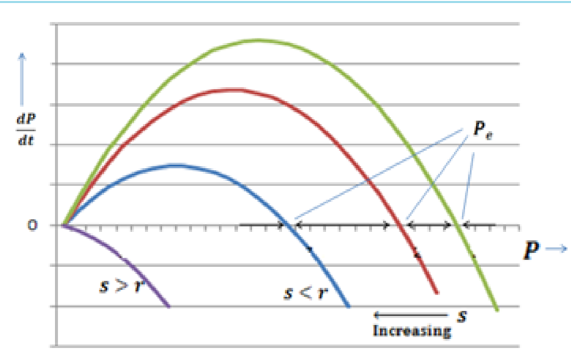

Fig. 3

We can consider the dynamic aspects of the process by determining the time scale of the recovery after harvesting. If $s = 0$ then, from (7), $P_e = K$, the recovery time t that is timescale of the reproductive growth from (6), we get (3).
But more realistically, the order of magnitude of the recovery time of P to its carrying capacity K after a small perturbation from K since, for $P(t)$ - K small and $P_e = K$, (12) shows

$$\frac{d(P - K)}{dt} \cong -r(P - K), \quad P(0) = P_0 \quad (13)$$

$$=> \quad P - K = (P_0 - K)\, e^{-rt} \qquad (14)$$

Recovery time to its carrying K after a small perturbation from K, for small $(P(t) - K)$,

$$t_R(s = 0) = \frac{1}{r}\ \ln\left(\frac{P_0 - K}{P - K}\right) \qquad (15)$$

If $s \neq 0$, with $0 < s < r$, then the recovery time in a harvesting situation, from (12) with the view of (9), is

$$t_R(s \neq 0) = \frac{1}{(r - s)}\ln\left[\left(1 - \frac{K}{P_0}\right)\left(1 - \frac{r}{s}\right)\right] \qquad (16)$$

And so

$$\frac{t_R(s \neq 0)}{t_R(s = 0)} = \frac{1}{\left(1 - \frac{s}{r}\right)} \; \frac{\ln\left[\left(1 - \frac{K}{P_0}\right)\left(1 - \frac{r}{s}\right)\right]}{\ln\left(\frac{P_0 - K}{P - K}\right)} \qquad (17)$$

Thus for a fixed r, a larger s increases the recovery time since $\frac{t_R(s\neq 0)}{t_R(s=0)}$ increases with s. When $s = r/2$, the value giving the maximum sustained yield P_M,

$$\frac{t_R\left(s = \frac{r}{2}\right)}{t_R(s = 0)} = 2 \;\; \frac{\ln(K - P_0) - \ln(P_0)}{\ln(K - P_0) - \ln(K - P)} \qquad (18)$$

The usual definition of a recovery time is the time to decrease a perturbation from equilibrium by a factor e. Then, on a linear basis,

$$t_R(s = 0) = \frac{1}{r} \;, t_R(s \neq 0) = \frac{1}{(r - s)}$$

$$=> \; t_R\left(s = \frac{r}{2}\right) = 2\, t_R(s = 0) \qquad (19)$$

Since it is the yield P_h that is recorded, if we solve (10) for s in terms of P_h we have

$$\frac{s}{r} = 1 \pm \sqrt{1 - \frac{P_h}{P_M}} \qquad (20)$$

Therefore

$$\frac{t_R(s \neq 0)}{t_R(s = 0)}$$

$$= \frac{2}{1 \pm \sqrt{1 - \frac{P_h}{P_M}}} \; \frac{\ln\left[\left(1 - \frac{K}{P_0}\right)\left(1 - \frac{r}{s}\right)\right]}{\ln\left(\frac{P_0 - K}{P - K}\right)} \qquad (21)$$

But on linear basis

$$\frac{t_R(s \neq 0)}{t_R(s = 0)} = \frac{2}{1 \pm \sqrt{1 - \frac{P_h}{P_M}}} \qquad (22)$$

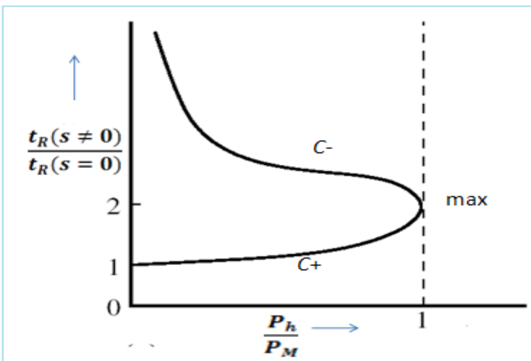

Fig. 4a

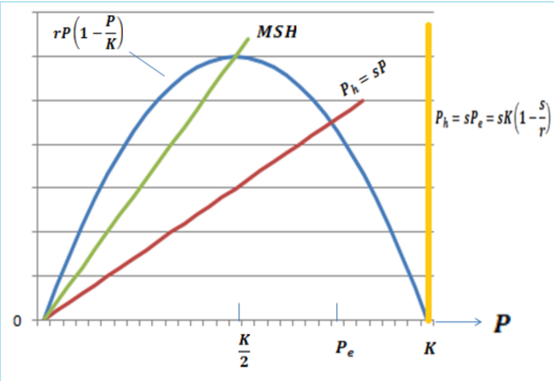

Fig. 4b

Suppose we start harvesting with a small effort s; then, as is clear from Figure (4b), the equilibrium population P_e is close to K and $P_e > K/2$, the equilibrium population for the maximum yield P_h. The recovery time ratio $\frac{t_R(s\neq 0)}{t_R(s=0)}$ from (19) is then approximately 1.

As we increase the harvesting rate s, the yield will increase until we are on branch C+. If s increases further, P_e decreases towards $K/2$, then we approach the value for the maximum sustained yield (MSH) P_M and finally we reach to the point max in Figure (4a) when $P_e = K/2$. As s is increased further such that $P_e < K/2$ then the recovery time is further increased but with a decreasing yield since we are now on the C- branch.

From this deterministic point of view, an optimal harvesting strategy could be determined. An effort s should be made in such a way that keeps the equilibrium population density $P_e > K/2$, but as close as possible to $K/2$ which gives the value for the maximum sustained yield. The closer to $K/2$, however, the situation becomes more delicate since we might inadvertently move onto branch C- in Figure (4a). At this stage, when P_e is close to $K/2$, a stochastic analysis should be carried out; this was done by Beddington and May [2] also which we have extended. Stochastic elements of course reduce the predictability of the catch. In fact, with this model, they decrease the average yield for a given effort s.

V. CONSTANT HARVESTING

When a constant number of individuals are harvested each year, a model studied by Brauer and Sanchez [3], our logistic model becomes

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P - B \cong f(P;r,K,B),$$
$$P(0) = P_0 \quad (23)$$

where r is a population growth rate constant, B is the number of individuals from the population that are harvested, P is the population at time t, and K is the carrying capacity of the environment.

The population P in constant harvesting can be estimated by the formula,

$$P = \frac{K}{2} + \frac{\alpha}{2}\left[\frac{1+\left\{\frac{2P_0-K-\alpha}{2P_0-K+\alpha}\right\}e^{\left(\frac{-r\alpha}{K}t\right)}}{1-\left\{\frac{2P_0-K-\alpha}{2P_0-K+\alpha}\right\}e^{\left(\frac{-r\alpha}{K}t\right)}}\right] \quad (24)$$

Where $\alpha = \sqrt{K^2 - \frac{4KB}{r}} \geq 0$

As $t \to \infty$, $P_{max} = \frac{K}{2} + \frac{\alpha}{2}$ (25)

This shows in figure 5 graphically,

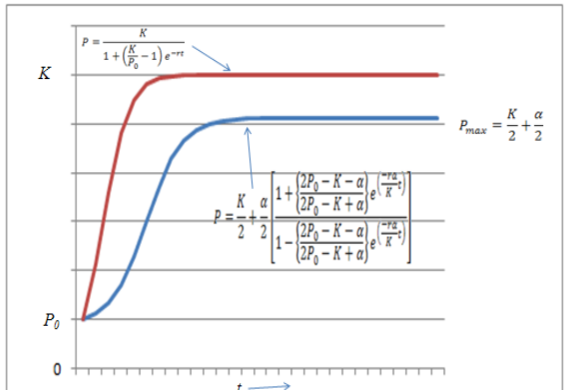

Fig. 5

And the time required getting a population P,

$$t = \frac{K}{r\alpha}\, ln\left[\frac{(2P_0-K-\alpha)(2P-K+\alpha)}{(2P_0-K+\alpha)(2P-K-\alpha)}\right]$$
$$\forall\ P \neq \frac{K+\alpha}{2} \quad (26)$$

Notice that in Equation (23), equilibrium is reached when the rate of change of the population is zero; that is when,

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P - B = 0 \quad (27)$$

Observe that the middle piece of this equation is quadratic with respect to the population size P. This implies that it is equal to zero for two specific populations.

$$\frac{r}{K}P^2 - rP + B = 0$$

OR

$$P^2 - KP + \frac{KB}{r} = 0$$

At first glance, we get

$$P = \frac{K \pm \alpha}{2}$$

Where $\alpha = \sqrt{K^2 - \frac{4KB}{r}} \geq 0$

These two values present an opportunity for two choices for harvest. We must, however, be sure that the sign of the derivative near these two values points us in the right direction. For example, in Equation (23), suppose those two populations' $p_1 = \frac{K-\alpha}{2}$ and $p_2 = \frac{K+\alpha}{2}$ produce two states of equilibrium, as shown by Fig. 6.

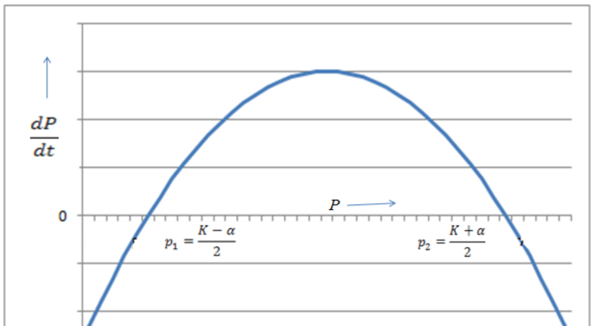

Fig. 6

In Figure 6, for values of P near, but below, p_2, notice that the derivative is positive, indicating that the population will increase to the equilibrium point p_2. For values of P near, but above, p_2 the derivative is negative, indicating that the population will decrease to the point p_2. This type of point is referred to as a *stable equilibrium*.

The same analysis on p_1 produces a different result. For example, the derivative is negative for values of P that are near, but below p_1, indicating that the population will reduce and move away from the equilibrium point. For values of P that are near, but above p_1, the derivative is positive, indicating that the population will increase beyond p_1. This type of equilibrium point is referred to as an *unstable equilibrium.*

The equilibrium states for the logistic growth harvested with a constant yield B in figure 7a and gives the graphical way of determining the steady states as B varies such that $0 < B < \frac{rK}{4}$. There are two positive steady states p_1 and p_2 which from Figure 7b are respectively unstable and stable. The growth rate $\frac{dP}{dt}$ in equation (23) changing as the yield B increases, shown in figure 7b.

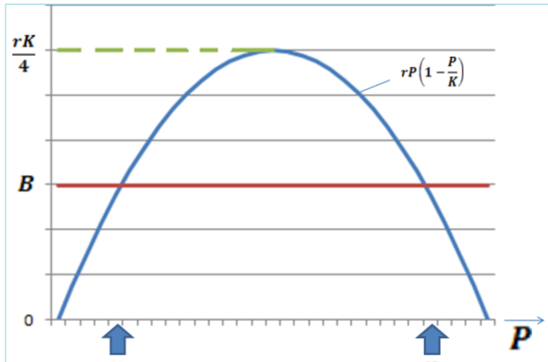

Fig. 7a

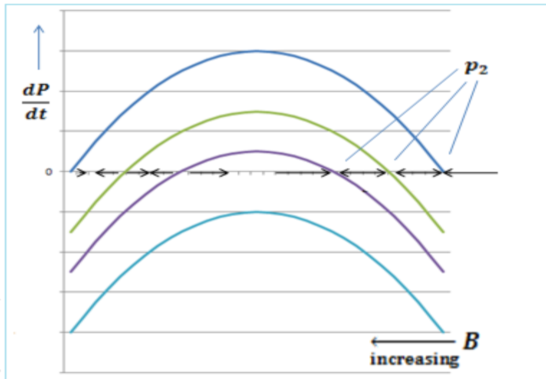

Fig. 7b

It is pertinent to manage resources so that the population stays relatively close to a stable equilibrium. In this situation, waiting until the population gets near a stable equilibrium such as p_2 is essential. To determine the maximum sustainable harvest, simply solve Equation (27) for B and then optimize the right-hand side of the equation. i.e. the maxima of a quadratic occurs at the vertex.

$$B = r\left(1 - \frac{P}{K}\right)P \qquad (28)$$

This gives the number of harvested population taken each year constantly.

Note that the right-hand side of Equation (28) is a quadratic function of P, which attains a maximum value when $P = K/2$. Thus, the maximum sustainable harvest for constant harvesting is found by substituting $P = K/2$ into Equation (28). That is,

$$P_M = MSH = B\left(at\ P = {}^{K}\!/_{2}\right)$$
$$= r\left(1 - \frac{{}^{K}\!/_{2}}{K}\right)\frac{K}{2} = \frac{rK}{4} \qquad (29)$$

For easy comparison with the constant harvesting model we evaluate the equivalent recovery time ratio $\frac{t_R(B\neq 0)}{t_R(B=0)}$. The recovery time $t_R(B \neq 0)$ is only relevant to the stable equilibrium p_2 which from (23) is

$$p_2 = \frac{K}{2}\left\{1 + \sqrt{1 - \frac{4B}{rK}}\right\}, \quad B < \frac{rK}{4} \qquad (30)$$

The linearised form of (23) is then

$$\frac{d(P - p_2)}{dt} \cong (P - p_2)\left(\frac{\partial f}{\partial P}\right)_{p_2}$$

$$= -(P - p_2)r\sqrt{1 - \frac{4B}{rK}} \qquad (31)$$

Now we can consider the dynamic aspects of the process by determining the time scale of the recovery after harvesting. If $B = 0$ then, from (30), $p_2 = K$, the recovery time *t* that is timescale of the reproductive growth from (26), we again get (3).

The order of magnitude of the recovery time of P to its carrying capacity K after a small perturbation from K since, for $P(t) - K$ small and $p_2 = K$, (31) shows

$$\frac{d(P - K)}{dt} \cong -r(P - K), \quad P(0) = P_0 \qquad (32)$$

$$=> \quad P - K = (P_0 - K)\, e^{-rt} \qquad (33)$$

Recovery time to its carrying K after a small perturbation from K, for small $(P(t) - K)$,

$$t_R(B = 0) = \frac{1}{r}\ \ln\left(\frac{P_0 - K}{P - K}\right) \qquad (34)$$

Here we have noticed that the solution of (13) & (31) with different consideration as $P_e = K$ & $p_2 = K$ respectively, is same shown (14) & (33). Similarly, the recovery time to its carrying capacity after a small perturbation from K, with different *s=0* & *B=0* is also same, shown (15) & (34) respectively.

If $B \neq 0$, then the recovery time in a harvesting situation, from (26), is

$$t_R(B \neq 0) = \frac{K}{r\alpha}\ \ln\left[\frac{(2P_0 - K - \alpha)(2P - K + \alpha)}{(2P_0 - K + \alpha)(2P - K - \alpha)}\right] \quad \forall\, P \neq \frac{K + \alpha}{2} \qquad (35)$$

And so

$$\frac{t_R(B \neq 0)}{t_R(B = 0)} = \frac{K}{\alpha}\ \frac{\ln\left[\frac{(2P_0 - K - \alpha)(2P - K + \alpha)}{(2P_0 - K + \alpha)(2P - K - \alpha)}\right]}{\ln\left(\frac{P_0 - K}{P - K}\right)} \qquad (36)$$

The usual definition of a recovery time is the time to decrease a perturbation from equilibrium by a factor *e*. Then, on a linear basis,

$$t_R(B = 0) = \frac{1}{r}\ , \qquad t_R(B \neq 0) = \frac{K}{r\alpha} \qquad (37)$$

Therefore,

$$\frac{t_R(B \neq 0)}{t_R(B = 0)} = \frac{1}{\sqrt{1 - \frac{4B}{rK}}}\ , \qquad (38)$$

Now using equation (29), we get

$$\frac{t_R(B \neq 0)}{t_R(B = 0)} = \frac{1}{\sqrt{1 - \frac{B}{P_M}}}\ , \quad P_M = \frac{rK}{4} \quad (39)$$

Which shows that $\frac{t_R(B\neq 0)}{t_R(B=0)} \to \infty$ as $B \to P_M$. This model is thus a much more sensitive one and, as a harvesting strategy, is not really adequate.

VI. DISCUSSION

The study concludes that a proportional harvesting rather than a constant yield harvesting strategy is potentially less disastrous. The derivations can be used to regulate to catch fishes with this simple model in the fishing laws. A more realistic model, on the lines described here, should take into account the economic costs of harvesting and other factors. This implies a feedback mechanism which can be a stabilising factor; Clark [6, 7, 8]. With the unpredictability of the real world it is probably essential to include feedback. Nevertheless such simple models can pose highly relevant ecological and long term financial factors which have to be considered in any more realistic and more sophisticated model.

This mathematical models can be used to (i) explain complex environmental processes and interactions, characterize the spatial and temporal variations, and predict the fate and transport of the contaminants; (ii) study risks existing in various resources-related activists and the associated socioeconomic and environmental effects under a variety of system conditions; (iii) generate sound decision alternatives for generating desired policies that target on more effective resources and environmental conservation.

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